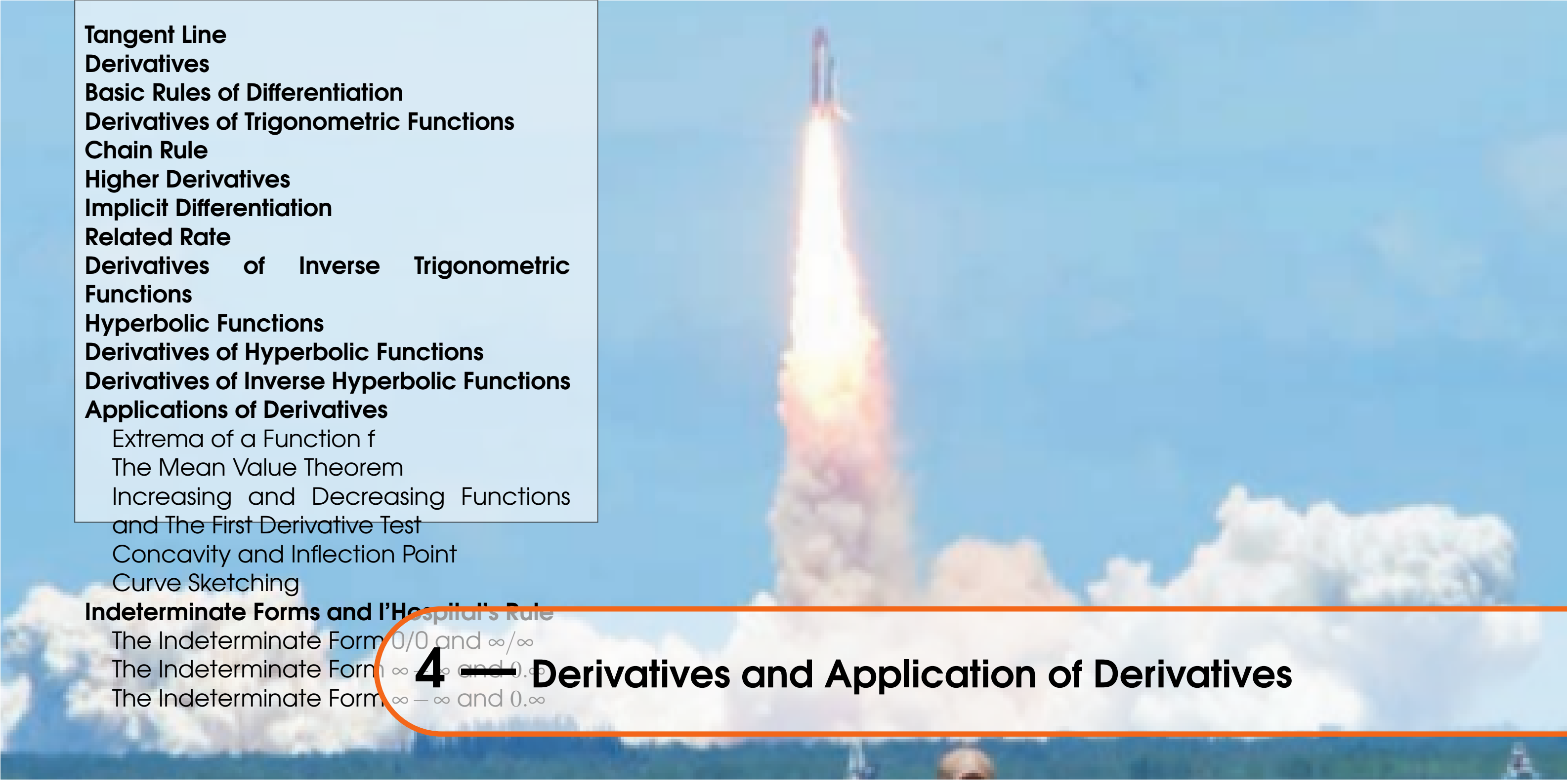


4 — Derivatives and Application of Derivatives

4.1 Tangent Line

Let P and Q be two distinct points on a curve, and consider the secant line passing through P and Q. (See Figure (4.1)). If we let Q move along the curve toward P, then the secant line rotates about P and approaches the fixed line T. We define T to be the tangent line at P on the curve. Let's make

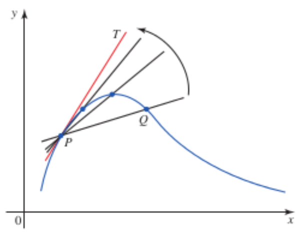
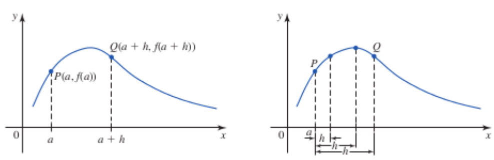


Figure 4.1: :As Q approaches P along the curve, the secant lines approach the tangent line T.

this notion more precise: Suppose that the curve is the graph of a function f defined by $y = f(x)$. (See Figure (4.2)). Let $P(a, f(a))$ be a point on the graph of f, and let Q be a point on the graph of f distinct from P. Then the x-coordinate of Q has the form $x = a+h$, where h is some appropriate nonzero number. If $h > 0$, then Q lies to the right of P; and if $h < 0$, then Q lies to the left of P. The corresponding y-coordinate of Q is $y = f(a+h)$. In other words, we can specify Q in the usual manner by writing $Q(a+h, f(a+h))$. Observe that we can make Q approach P along the graph of f by letting h approach 0. This situation is illustrated in Figure (4.2)b. Next, using the formula



(a) The points $P(a, f(a))$ and $Q(a+h, f(a+h))$ (b) As h approaches 0, Q approaches P.

Figure 4.2:

for the slope of a line, we can write the slope of the secant line passing through $P(a, f(a))$ and $Q(a+h, f(a+h))$ as:

$$M = \frac{f(a+h)-f(a)}{(a+h)-a} = \frac{f(a+h)-f(a)}{h} \tag{4.1}$$

The expression on the right-hand side of Equation 4.1 is called a difference quotient. As we observed earlier, if we let h approach 0, then Q approaches P and the secant line passing through P and Q approaches the tangent line T. This suggests that if the tangent line does exist at P, then its slope m should be the limit of M obtained by letting h approach zero. This leads to the following definition.

Definition 4.1.1 The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with slope

$$m = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a} \tag{4.2}$$

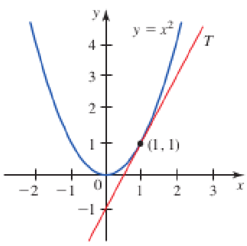
provided that this limit exists.
We can find an equation of the tangent line at P by using the point-slope form of an equation of a line. Thus,

$$y = m(x-a) + f(a)$$

Definition 4.1.2 The normal line to a curve at a given point is the line perpendicular to the tangent line at that Point.

Example 4.1 Find an equation of the tangent line to the parabola $y = x^2$ at the point $(1,1)$

Solution: To find the slope of the tangent line at the point $P(1,1)$, we use Equation 4.1 with $a = 1$, obtaining



$$\begin{aligned} m &= \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{h\to 0} \frac{(1+h)^2-(1)^2}{h} \\ &= \lim_{h\to 0} \frac{(1+2h+h^2)-(1)}{h} \\ &= \lim_{h\to 0} \frac{h^2+2h}{h} = \lim_{h\to 0}(h+2) = 2 \end{aligned}$$

To find an equation of the tangent line, we use the point-slope form of an equation of a line to obtain

$$\begin{aligned} y - f(a) = m(x-a) \quad &\Rightarrow \quad y - 1 = 2(x-1) \\ &\Rightarrow \quad y = 2x - 1 \end{aligned}$$

■ **Example 4.2** Find an equation of the normal line to the curve $y = \sqrt{x-3}$ which is parallel to the line $6x + 3y - 4 = 0$. ■

Solution: Let l be the given line. To find the slope of l, we write its equation in the slope-intercept form, which is

$$y = -2x + \frac{4}{3}$$

Therefore, the slope of l is -2 and the slope of the desired normal line is also -2 because the two lines are parallel.
Now we need to find the slope of the given curve at any point $(a, f(a))$, we have

$$\begin{aligned} m &= \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \frac{\sqrt{x-3}-\sqrt{a-3}}{x-a} \\ &= \lim_{x\to a} \left(\frac{\sqrt{x-3}-\sqrt{a-3}}{x-a}\right)\left(\frac{\sqrt{x-3}+\sqrt{a-3}}{\sqrt{x-3}+\sqrt{a-3}}\right) \\ &= \lim_{x\to a} \frac{(x-3)-(a-3)}{(x-a)(\sqrt{x-3}+\sqrt{a-3})} \\ &= \lim_{x\to a} \frac{1}{\sqrt{x-3}+\sqrt{a-3}} = \frac{1}{2\sqrt{a-3}} \end{aligned}$$

Because the normal line at a point is perpendicular to the tangent line at that point, the product of their slopes is -1. Hence, the slope of the normal line at $(a, f(a))$ is given by $-2\sqrt{a-3}$. Since the slope of the desired line is -2, so we solve the equation

$$-2\sqrt{a-3} = -2 \quad \Rightarrow \quad a = 4$$

Therefore, the desired line is the line through point $(4,1)$ on the curve and has a slope of -2. Using the point-slope form of an equation of a line we obtain,

$$y = -2(x-4)+1 \Rightarrow y = -2x+9$$

4.2 Derivatives

Definition 4.2.1 The derivative of a function f with respect to x is the function f' defined by the rule

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \tag{4.3}$$

The domain of f' consists of all values of x for which the limit exists.

R If we let $x = a + h$ then $h = x - a$ hence $h \to 0$ iff $x \to a$.
Therefore,

$$f'(x) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}$$

Two interpretations of the derivative follow.

1. **Geometric Interpretation of the Derivative:** The derivative f' of a function f is a measure of the slope of the tangent line to the graph of f at any point $(x, f(x))$, provided that the derivative exists.
2. **Physical Interpretation of the Derivative:** The derivative f' of a function f measures the instantaneous rate of change of f at x. (See Figure 4.3.)
Note:
1. The value of the derivative of f at a is denoted by $f'(a)$
2. The tangent line to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ whose slope is equal to $f'(a)$.

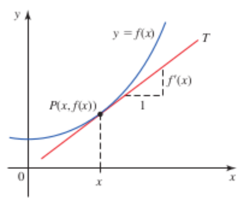


Figure 4.3: $f'(x)$ is the slope of T at P; $f(x)$ is changing at the rate of $f'(x)$ units per unit change in x at x.

■ **Example 4.3** Let $f(x) = \sqrt{5-x^2}$. Find

1. $f'(x)$ and determine the domain of the derivative function
2. an equation of the tangent line to the graph of $f(x)$ at $x = -1$

■

Solution:

1. From the above difinition we have,

$$\begin{aligned}
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{5-(x+h)^2}-\sqrt{5-x^2}}{h}\\
&= \lim_{h\to 0}\left(\frac{\sqrt{5-(x^2+2xh+h^2)}-\sqrt{5-x^2}}{h}\right)\left(\frac{\sqrt{5-(x^2+2xh+h^2)}+\sqrt{5-x^2}}{\sqrt{5-(x^2+2xh+h^2)}+\sqrt{5-x^2}}\right)\\
&= \lim_{h\to 0}\frac{(5-x^2-2xh-h^2)-(5-x^2)}{h(\sqrt{5-(x^2+2xh+h^2)}+\sqrt{5-x^2})}\\
&= \lim_{h\to 0}\frac{-2xh-h^2}{h(\sqrt{5-(x^2+2xh+h^2)}+\sqrt{5-x^2})}\\
&= \lim_{h\to 0}\frac{-2x-h}{\sqrt{5-(x^2+2xh+h^2)}+\sqrt{5-x^2}} = \frac{-2x}{\sqrt{5-x^2}+\sqrt{5-x^2}}\\
&= \frac{-x}{\sqrt{5-x^2}}
\end{aligned}$$

Since $f'(x)$ exist if $-\sqrt{5} < x < \sqrt{5}$, then the domain of $f'(x)$ is $(-\sqrt{5}, \sqrt{5})$

2. The slope m of the tangent line to the graph of $f(x)$ at $x = -1$ is

$$f'(-1) = \frac{-(-1)}{\sqrt{5-(-1)^2}} = \frac{1}{2}$$

■ **Example 4.4** Let $y = \dfrac{x-2}{-x+1}$. Find $\dfrac{dy}{dx}$ ■

Solution:

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\dfrac{(x+h)-2}{-(x+h)+1}-\dfrac{x-2}{-x+1}}{h}\\
&= \lim_{h\to 0}\frac{(x+h-2)(-x+1)-(x-2)(-x-h+1)}{h(-x-h+1)(-x+1)}\\
&= \lim_{h\to 0}\frac{-x^2+x+h-xh-2+2x-[-x^2+2x-xh+2h+x-2]}{h(-x-h+1)(-x+1)}
\end{aligned}$$

$$= \lim_{h\to 0} \frac{-h}{h(-x-h+1)(-x+1)}$$
$$= \lim_{h\to 0} \frac{-1}{(-x-h+1)(-x+1)} = \frac{-1}{(x-1)^2}$$

Example 4.5 Calculate the instantaneous velocity at time $t = 5$ of an automobile whose position at time t seconds is given by $g(t) = t^3 + 4t^2 + 10$ feet. ■

Solution: We know that the required instantaneous velocity is $g'(5)$. We calculate

$$\begin{aligned} g'(5) &= \lim_{h\to 0} \frac{g(5+h)-g(5)}{h} \\ &= \lim_{h\to 0} \frac{(5+h)^3+4(5+h)^2+10-\left[5^3+4.5^2+10\right]}{h} \\ &= \lim_{h\to 0} \frac{115h+19h^2+h^3}{h} \\ &= \lim_{h\to 0} 115+19h+h^2 = 115 \end{aligned}$$

We conclude that the instantaneous velocity of the moving body at time $t = 5$ is $g'(5) = 115$ ft/sec.

Example 4.6 A rubber balloon is losing air steadily. At time t minutes the balloon contains $75-10t^2+t$ cubic inches of air. What is the rate of loss of air in the balloon at time $t = 1$? ■

Solution: Let $f(t) = 75-10t^2+t$. We calculate

$$\begin{aligned} f'(1) &= \lim_{h\to 0} \frac{f(1+h)-f(1)}{h} \\ &= \lim_{h\to 0} \frac{-19h-10h^2}{h} \\ &= \lim_{h\to 0} (-19h-10h^2) = -19 \end{aligned}$$

In conclusion, the rate of air loss in the balloon at time t = 1 is $f'(1) = -19\ ft^3/sec$. Observe that the negative sign in this answer indicates that the change is negative, i.e., that the quantity is decreasing.

Other Notation:

If we denote the dependent variable by y so that $y = f(x)$, then some common alternative notations for the derivative are as follows:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$ The value of the derivative at a is denoted by $\frac{dy}{dx}|_{x=a}$

Definition 4.2.2 A function f is differentiable at a if $f'(a)$ exists.

Definition 4.2.3 A function is said to be differentiable if it is differentiable at every number in its domain.

Example 4.7 Show that the function $f(x) = |x|$ is differentiable everywhere except at 0. ■

Solution: To prove that f is not differentiable at 0, we will show that $f'(0)$ does not exist by demonstrating that the one-sided limits of the quotient

$$\frac{f(0+h)-f(0)}{h} = \frac{|h|-0}{h} = \frac{|h|}{h}$$

as h approaches 0 are not equal.

First, suppose $h > 0$. Then $|h| = h$, so $\lim_{h\to 0} \frac{|h|}{h} = \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1$

Next, if $h < 0$, then $|h| = -h$, and therefore,

$$\lim_{h\to 0} \frac{|h|}{h} = \lim_{h\to 0} \frac{-h}{h} = \lim_{h\to 0} -1 = -1$$

Therefore

$$f'(0) = \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} = \lim_{h\to 0} \frac{|h|}{h}$$

does not exist, and f is not differentiable at 0.

To show that f is differentiable at all other numbers, we rewrite f(x) in the form

$$f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -x & if\, x < 0 \end{cases}$$

and then differentiate f(x) to obtain

$$f'(x) = \begin{cases} 1 & \text{if } x > 0 \\ -1 & if\, x < 0 \end{cases}$$

Theorem 4.2.1 If a function f is differentiable at a then f is continuous at a

Proof. If x is in the domain of f and $x \neq 0$ then we can write

$$f(x) - f(a) = \frac{f(x)-f(a)}{x-a}(x-a) \text{ We have}$$

$$\lim_{x\to a}[f(x)-f(a)] = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}(x-a) = \lim_{x\to a} \frac{f(x)-f(a)}{x-a}.\lim_{x\to a}(x-a) = f'(x).0 = 0$$

So,

$$\lim_{x\to a} f(x) = \lim_{x\to a}[f(a)+(f(x)-f(a))] = \lim_{x\to a} f(a) + \lim_{x\to a}[f(x)-f(a)] = f(a)+0 = f(a)$$

and this shows that f is continuous at a, as asserted. ■

4.3 Basic Rules of Differentiation

Theorem 4.3.1 — Derivative of a Constant Function.
If c is a constant, then

$$\frac{d}{dx}(c) = 0$$

Theorem 4.3.2 — The Power Rule.
If n is any real number and $f(x) = x^n$, then

$$f'(x) = \frac{d}{dx}(x^n) = nx^{n-1}$$

Example 4.8 If

1. $f(x) = x^{12}$, then $f'(x) = \frac{d}{dx}(x^{12}) = 12x^{11}$
2. $f(x) = \frac{1}{x^5}$ then $f'(x) = \frac{d}{dx}(\frac{1}{x^5}) = -5x^{-5-1} = \frac{-5}{x^6}$

■

Theorem 4.3.3 — The Constant Multiple Rule.
If f is a differentiable function and c is a constant, then
$$\frac{d}{dx}[cf(x)] = cf'(x)$$

■ **Example 4.9** If $f(x) = -6x^2$, then $f'(x) = -12x$ ■

Theorem 4.3.4 — The Sum Rule.
If f and g are differentiable functions, then
$$\frac{d}{dx}[f(x) \pm g(x)] = \frac{d}{dx}f(x) \pm \frac{d}{dx}g(x) = f'(x) + g'(x)$$

■ **Example 4.10** Find the derivative of $f(x) = 6x^8 + 3x^4 - 4x^3 + x^2 - 5x + 9$ ■

Solution:

$$\begin{aligned}
f'(x) &= \frac{d}{dx}[6x^8 + 3x^4 - 4x^3 + x^2 - 5x + 9] \\
&= 6\frac{d}{dx}(x^8) + 3\frac{d}{dx}(x^4 - 4\frac{d}{dx}(x^3) + \frac{d}{dx}(x^2) - 5\frac{d}{dx}(x) + \frac{d}{dx}(9) \\
&= 48x^7 + 12x^3 - 12x^2 + 2x - 5
\end{aligned}$$

Theorem 4.3.5 — The Product Rule.
If f and g are differentiable functions, then
$$\frac{d}{dx}[f(x)g(x)] = g(x)\frac{d}{dx}f(x) + f(x)\frac{d}{dx}g(x) = f'(x)g(x) + g'(x)f(x)$$

■ **Example 4.11** Find the derivative of $f(x) = (x^2 - 1)(\sqrt{x} - 2x)$ ■

Solution:

$$\begin{aligned}
f'(x) &= (\sqrt{x} - 2x)\frac{d}{dx}(x^2 - 1) + (x^2 - 1)\frac{d}{dx}(\sqrt{x} - 2x) \\
&= (\sqrt{x} - 2x)(2x) + (x^2 - 1)\left[\frac{d}{dx}(\sqrt{x}) - \frac{d}{dx}(2x)\right] \\
&= 2x\sqrt{x} - 4x^2 + (x^2 - 1)\left[\frac{1}{2\sqrt{x}} - 2\right] \\
&= 2x\sqrt{x} - 4x^2 + \frac{x^2}{2\sqrt{x}} - 2x^2 - \frac{1}{2\sqrt{x}} + 2 \\
&= 2x\sqrt{x} - 6x^2 + \frac{x^2 - 1}{2\sqrt{x}} + 2
\end{aligned}$$

■ **Example 4.12** Find the derivative of $f(x) = (xe^x + x)(x^3 - \ln x)$ ■

Theorem 4.3.6 — The Quotient Rule.
If f and g are differentiable functions and $g(x) \neq 0$, then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - g'(x)f(x)}{(g(x))^2}$$

■ **Example 4.13** Find the derivative of $f(x) = \dfrac{x^2 - x}{x^3 + 1}$ ■

Solution:

$$\begin{aligned} f'(x) &= \frac{d}{dx}\left(\frac{x^2-x}{x^3+1}\right) = \frac{(x^3+1)\frac{d}{dx}(x^2-x) - (x^2-x)\frac{d}{dx}(x^3+1)}{(x^3+1)^2} \\ &= \frac{(x^3+1)(2x-1) - (x^2-x)(3x^2)}{(x^3+1)^2} = \frac{2x^4 - x^3 + 2x - 1) - (3x^4 - 3x^3)}{(x^3+1)^2} \\ &= \frac{5x^4 + 2x^3 + 2x - 1}{(x^3+1)^2} \end{aligned}$$

4.4 Derivatives of Trigonometric Functions

Derivatives of Sines and Cosines

Theorem 4.4.1 Derivatives of $\sin x$

$$\frac{d}{dx}(\sin x) = cos x$$

Let $f(x) = \sin x$

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{\sin(x+h) - \sin x}{h} \\ &= \lim_{h\to 0}\frac{\sin x \cos h + \cos x \sin h - \sin x}{h} = \lim_{h\to 0}\frac{\sin x\,(\cos h - 1) + \cos x \sin h}{h} \\ &= \sin x \lim_{h\to 0}\frac{(\cos h - 1)}{h} + \cos x \lim_{h\to 0}\frac{\sin h}{h} \\ &= \sin x.0 + \cos x.1 \\ &= \cos x \end{aligned}$$

Theorem 4.4.2 — Derivative of $\cos x$.

$$\frac{d}{dx}\cos x = \sin x$$

Theorem 4.4.3 — Rules for Differentiating Trigonometric Functions.

$a.\ \dfrac{d}{dx}\sin x = \cos x$ $\quad b.\ \dfrac{d}{dx}\cos x = \sin x$ $\quad c.\ \dfrac{d}{dx}(\sec x) = \sec x \tan x$

$d.\ \dfrac{d}{dx}(\csc x) = -\csc x \cot x$ $\quad e.\ \dfrac{d}{dx}(\tan x) = \sec^2 x$ $\quad f.\ \dfrac{d}{dx}(\cot x) = -\csc^2 x$

■ **Example 4.14** Differentiate $y = (\sec x)(x - \tan x)$. ■

■ **Example 4.15** Find the derivative of

(a) $f(x) = (\sin x + x)(x^3 - \ln x)$

(b) $h(x) = \dfrac{\sin x}{1 - 2\cos x}$

(c) $g(x) = \left(\dfrac{e^x + x\sin x}{\tan x}\right)$

■

Exercise 4.1 1. Differentiate $\dfrac{\sec x}{1 + \tan x}$. For what values of x does the graph of f have a horizontal tangent?

2. $y = (1 + \cos 3x^2)^4$. Find $\dfrac{dy}{dx}$

■

4.5 Chain Rule

Theorem 4.5.1 — The chain Rule.
If g is differentiable at x and f is differentiable at $g(x)$, then the composite function $F = f \circ g$ defined by $F(x) = f(g(x))$ is differentiable at x and F' is given by the product

$$F'(x) = f'(g(x)).g'(x)$$

Also, if we write $u = g(x)$ and $y = f(u) = f(g(x))$, then $\dfrac{dy}{dx} = \dfrac{dy}{du}\dfrac{du}{dx}$

■ **Example 4.16** Find $F'(x)$ if $F(x) = \sqrt{x^2 + 1}$. ■

■ **Example 4.17** Find the derivative of

(a) $f(x) = \sin(x^3 - x^2)$

(b) $\ln\left(\dfrac{x^2}{x - 2}\right)$

(c) $g(x) = \cos\left(\dfrac{2x}{1 + x}\right)$

■

■ **Example 4.18** Find $\dfrac{dy}{dx}$ if $y = u^3 - u^2 + u + 1$ and $u = x^3 + 1$ ■

Solution:

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} = (3u^2 - 2u + 1)(3x^2) \\ &= 3x^2(3x^6 + 4x^3 + 2) \end{aligned}$$

4.6 Higher Derivatives

The derivative f' of a function f is itself a function. As such, we may consider differentiating the function f'. The derivative of f', if it exists, is denoted by f'' and is called the second derivative of f . Continuing in this fashion, we are led to the third, fourth, fifth, and higher-order derivatives of f, whenever they exist. Notations for the first, second, third, and in general, the n^{th} derivative of f are

$$f', f'', f''', \ldots, f^{(n)}$$

or

$$\frac{d}{dx}[f(x)],\ \frac{d^2}{dx^2}[f(x)],\ \ldots,\ \frac{d^n}{dx^n}[f(x)]$$

respectively.

■ **Example 4.19** Find the third derivative of $y = \frac{1}{x} \implies y''' = -\frac{6}{x^4}$ ■

■ **Example 4.20** The position of a particle moving along a straight line is given by

$$s = f(t) = 2t^3 - 15t^2 + 24t, \quad t \geq 0$$

where t is measured in seconds and s in feet.

(a) Find an expression giving the velocity of the particle at any time t. What are the velocity and speed of the particle when $t = 2$?
(b) Determine the position of the particle when it is stationary.
(c) Find the acceleration function of the particle. What is the acceleration of the particle when $t = 2$?
(d) When is the acceleration zero? Positive? Negative?

■

Solution:

(a) The required velocity of the particle is given by

$$v(t) = \frac{ds}{dt} = f'(t) = 6t^2 - 30t + 24 = 6(t-1)(t-4)$$

The velocity of the particle when $t = 2$ is

$$v(2) = 6(2-1)(2-4) = -12$$

The speed of the particle when $t = 2$ is $|v(2)| = 12$ ft/sec. In short, the particle is moving in the negative direction at a speed of 12 ft/sec.

(b) The particle is stationary when its velocity is equal to zero. Setting $v(t) = 0$ gives

$$v(t) = 6(t-1)(t-4) = 0$$

and we see that the particle is stationary at $t = 1$ and $t = 4$.
Its position at $t = 1$ is given by

$$f(1) = 2(1)^3 - 15(1)^2 + 24(1) = 11$$

Its position at $t = 4$ is given by

$$f(4) = 2(4)^3 - 15(4)^2 + 24(4) = -16$$

(c) $a(t) = \frac{dv}{dt} = f''(t) = 12t - 30$

In particular, the acceleration of the particle when $t = 2$ is

$$a(2) = 12(2) - 30 = -6$$

The particle is decelerating at 6 ft/sec 2 when $t = 2$.

(d) The acceleration of the particle is zero when $a(t) = 0$, or

$$12t - 30 = 0 \implies t = \frac{5}{2}.$$

Since $2t - 5 < 0$ when $t < \frac{5}{2}$ and $2t - 5 > 0$ when$t > \frac{5}{2}$, we also conclude that the acceleration is negative for $0 < t < \frac{5}{2}$ and positive for $t > \frac{5}{2}$.

4.7 Implicit Differentiation

Suppose that a function $y = f(x)$ is defined implicitly via an equation in x and y. To compute $\frac{dy}{dx}$:

1. Differentiate both sides of the equation with respect to x. Make sure that the derivative of any term involving y includes the factor $\frac{dy}{dx}$.
2. Solve the resulting equation for $\frac{dy}{dx}$ in terms of x and y.

Example 4.21 Find $\frac{dy}{dx}$ if $2x^2y^2 - 3x^3 + 5y^3 + 6xy^2 = 5$

Solution: $\frac{dy}{dx} = \frac{9x^2 - 6y^2 - 4xy^2}{4xy^2 + 15y^2 + 12xy}$

Example 4.22 Find $\frac{dy}{dx}$ at $(\frac{\pi}{2}, \pi)$ if $x\sin y - y\cos 2x = 2x$

Solution: $\frac{dy}{dx} = \frac{4}{2-\pi}$

Example 4.23 Find an equation of the tangent line to the bifolium

$$4x^4 + 8x^2y^2 - 25x^2y + 4y^4 = 0$$

at the point $(2,1)$.

Solution: The slope of the tangent line to the bifolium at any point (x,y) is given by $\frac{dy}{dx}$. To compute $\frac{dy}{dx}$, we differentiate both sides of the equation with respect to x to obtain

$$\frac{d}{dx}\left(4x^4 + 8x^2y^2 - 25x^2y + 4y^4\right) = \frac{d}{dx}(0)$$

$$\Longrightarrow \quad 16x^3 + 16x^2y\frac{dy}{dx} + 16xy^2 - 25x^2\frac{dy}{dx} - 50xy + 16y^3\frac{dy}{dx} = 0$$

By substituting $x = 2$ and $y = 1$ into the last equation, we obtain

$$\frac{dy}{dx} = 3$$

Using the slope-intercept form for an equation of a line, we see that an equation of the tangent line is

$$y = 3x - 5$$

Example 4.24 Find $y'(x)$ implicitly for $y^4 + x^4 = 16$. Then find the value of y'' at the point $(-2,0)$.

Solution: $4y^3y' + 4x^3 = 0 \implies y' = -\frac{x^3}{y^3}$

To find y'' we differentiate this expression for y' using the Quotient Rule and remembering that y is a function of x:

$$\begin{aligned} y'' &= \frac{d}{dx}\left(-\frac{x^3}{y^3}\right) = -\frac{3x^2y^3 - 3x^3y^2y'}{y^6} = -\frac{3x^2y^3 - 3x^3y^2\left(-\frac{x^3}{y^3}\right)}{y^6} \\ &= -\frac{3x^2(y^4 + x^4)}{y^7} = -\frac{3x^2(16)}{y^7} = -\frac{48x^2}{y^7} \end{aligned}$$

4.8 Related Rate

In a related rates problem the idea is to compute the rate of change of one quantity in terms of the rate of change of another quantity (which may be more easily measured). The procedure is to find an equation that relates the two quantities and then use the Chain Rule to differentiate both sides with respect to time.

Guidelines for Solving a Related Rates Problem

(a) Draw a diagram, and label the variable quantities.
(b) Write down the given values of the variables and their rates of change with respect to t.
(c) Find an equation that relates the variables.
(d) Differentiate both sides of this equation implicitly with respect to t.
(e) Replace the variables and derivative in the resulting equation by the values found in Step 2, and solve this equation for the required rate of change.

Example 4.25 At a distance of 12,000 feet from the launch site, a spectator is observing a rocket being launched vertically. What is the speed of the rocket at the instant when the distance of the rocket from the spectator is 13,000 ft and is increasing at the rate of 480 ft/sec? ■

Solution:

Step 1 Let $y =$ the altitude of the rocket and $z =$ the distance of the rocket from the spectator at any time t. (See Figure 4.4.)

Step 2 We are given that at a certain instant of time

$$z = 13,000 \text{ and } \frac{dz}{dt} = 480,\ \frac{dy}{dt} = ?$$

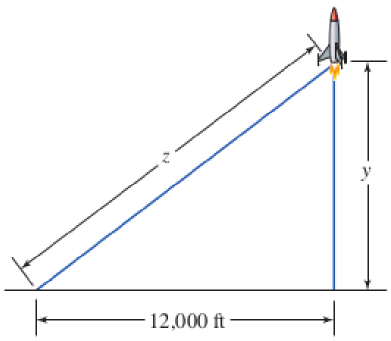


Figure 4.4:

Step 3 Applying the Pythagorean Theorem to the right triangle in Figure 4.4, we find that

$$z^2 = y^2 + (12,000)^2 \tag{4.4}$$

Step 3 Differentiating Equation (4.4) implicitly with respect to t, we obtain

$$2z\frac{dz}{dt} = 2y\frac{dy}{dt} \tag{4.5}$$

Step 3 Using Equation (4.4) we see that if $z = 13,000$, then

$$y = \sqrt{13000^2 - 12000^2} = 5000$$

Finally, substituting $z = 13,000,\ y = 5000$, and $\frac{dz}{dt} = 480$ in Equation (4.5), we find

$$(13000)(480) = 5000\frac{dy}{dt} \implies \frac{dy}{dt} = 1248 \tag{4.6}$$

Therefore, the rocket is rising at the rate of $1248 ft/sec$.

■ **Example 4.26** A 10-foot ladder leans against the side of a building. If the top of the ladder begins to slide down the wall at the rate of 2 ft/sec, how fast is the bottom of the ladder sliding away from the wall when the top of the ladder is 8 feet off the ground? ■

Solution: Let $y =$ the height of the top of the ladder and $x =$ the distance from the wall to the bottom of the ladder. Since the ladder is sliding down the wall at the rate of 2 ft/sec, we have that

$$\frac{dy}{dt} = -2$$

By the Pythagorean Theorem, we have

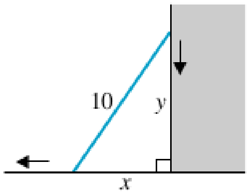


Figure 4.5:

$$x^2 + y^2 = 100$$

Differentiating both sides of this equation with respect to time gives us

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0 \implies \frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt}$$

Since the height above ground of the top of the ladder at the point in question is 8 feet, we have that $y = 8$ and from the Pythagorean Theorem, we get

$$100 = x^2 + 8^2,$$

so that $x = 6$. We now have that at the point in question,

$$\frac{dx}{dt} = -\frac{y}{x}\frac{dy}{dt} = -\frac{8}{6}(-2) = \frac{8}{3}$$

So, the bottom of the ladder is sliding away from the building at the rate of $\frac{8}{3} ft/sec$.

■ **Example 4.27** A water tank has the shape of an inverted circular cone with base radius $2m$ and height $4m$. If water is being pumped into the tank at a rate of $2m^3/min$, find the rate at which the water level is rising when the water is $3m$ deep. ■

Solution: Let V, r, and h be the volume of the water, the radius of the surface, and the height of the water at time t, where t is measured in minutes.

We are given that $\frac{dV}{dt} = 2m^3/min$ and we are asked to find $\frac{dh}{dt}$ when h is $3m$. The quantities V and h

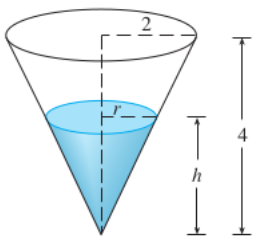


Figure 4.6:

are related by the equation

$$V = \frac{1}{3}\pi r^2 h$$

To do this, we use similar triangles and deduce that

$$\frac{r}{h} = \frac{2}{4} \implies r = \frac{1}{2}h$$

Substituting this value of r into the expression for V, we obtain

$$V = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h = \frac{1}{12}\pi h^3$$

Now we can differentiate each side with respect to t:

$$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt} \implies \frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$$

Substituting $h = 3m$ and $\frac{dV}{dt} = 2m^3/min$, we have

$$\frac{dh}{dt} = \frac{4}{\pi(3)^2}.2 = \frac{8}{9\pi}$$

Example 4.28 A boat is being pulled toward a dock by means of a rope attached to the front tip of the bow. Initially there are 30 feet of rope out and the rope is taught and being reeled in by a circular device the top of which is 10 feet higher than the point where the rope is attached to the boat. This circular device has a radius of 1 foot and turns at the rate of one revolution every pi seconds. How fast is the boat moving along the water when there are 15 feet of rope out? ■

Let

y = amount of rope out

x = distance the boat is from the base of the dock

The circumference of the circular device is 2π ft. so 2π feet of rope are pulled in every π seconds. Thus y is decreasing by $\frac{2\pi}{\pi} = 2$ ft/sec. i.e.,

$$\frac{dy}{dt} = -2$$

We want to find $\frac{dx}{dt}$ when y = 15 (and so $x = \sqrt{(15^2 - 10^2)} = 5\sqrt{5}$ by the Pythagorean Theorem).

$$x^2 + 10^2 = y^2 \Rightarrow 2x\frac{dx}{dt} = 2y\frac{dy}{dt}$$

$$\Rightarrow \frac{dx}{dt} = \frac{y}{x}\frac{dy}{dt} = -2\frac{15}{5\sqrt{5}} = \frac{-6}{\sqrt{5}} ft/sec$$

when $y = 15ft$ (negative because x is decreasing).

What happens when the boat gets very close to the base of the dock?

Example 4.29 At 1:00 P.M., ship A is 25 mile due south of ship B. If ship A is sailing west at a rate of 16 mi/hr and ship B is sailing south at a rate of 20 mi/hr, find the rate at which the distance between the ships is changing at 1:30 P.M. ■

Let x and y denote the miles covered by ships A and B, respectively, in t hours after 1:00 P.M. We then have the situation sketched in Figure 4.7, where P and Q are their respective positions at 1: 00 P.M. and z is the distance between the ships at time t.

By the Pythagorean Theorem,

$$z^2 = x^2 + (25 - y)^2$$

Differentiating with respect to t, we obtain

$$2z\frac{dz}{dt} = 2x\frac{dx}{dt} - 2(25 - y)\frac{dy}{dt} \implies z\frac{dz}{dt} = x\frac{dx}{dt} - (25 - y)\frac{dy}{dt}$$

It is given that

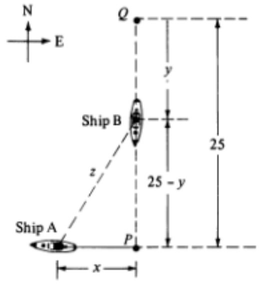


Figure 4.7:

$$\frac{dx}{dt} = 16mi/hr \quad \text{and} \quad \frac{dy}{dt} = 20mi/hr$$

Our objective is to find $\frac{dz}{dt}$.

At 1:30 P.M. the ships have traveled for half an hour and we have $x = 8, y = 10, 25 - y = 15$, and, therefore, $z^2 = 64 + 225 = 289, or z = 17$.

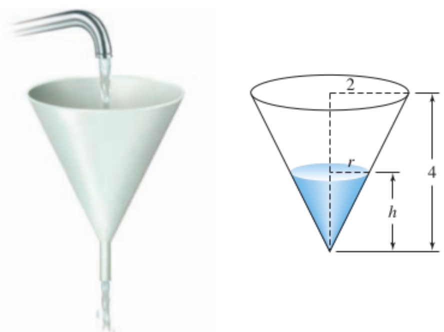
Substituting in the equation involving $\frac{dz}{dt}$ gives us

$$17\frac{dz}{dt} = 8(16) - 20(25 - 10) \implies \frac{dz}{dt} = -\frac{172}{17} \approx -10.12mi/hr$$

The negative sign indicates that the distance between the ships is decreasing at 1: 30 P.M.

Example 4.30 Water is poured into a conical funnel at the constant rate of $1in.^3/sec$ and flows out at the rate of $12in.^3/sec$ (Figure 4.8a). The funnel is a right circular cone with a height of $4in.$ and a radius of $2in.$ at the base (Figure 4.8b). How fast is the water level changing when the water is 2 in. high?

Solution: Let V = the volume of the water in the funnel
h = the height of the water in the funnel
and r = the radius of the surface of the water in the funnel We are given that



(**a**) Water is poured into a conical funnel.

(**b**) We want to find the rate at which the water level is rising when $h = 2$.

Figure 4.8:

$$\frac{dV}{dt} = 1 - \frac{1}{2} = \frac{1}{2} \quad \text{(Rate of flow in minus rate of flow out)}$$

and are asked to find $\frac{dh}{dt}$ when $h = 2$.

The volume of water in the funnel is equal to the volume of the shaded cone in Figure 4.8b. Thus,

$$V = \frac{1}{3}\pi r^2 h$$

To do this, we use similar triangles and deduce that

$$\frac{r}{h} = \frac{2}{4} \implies r = \frac{1}{2}h$$

Substituting this value of r into the expression for V, we obtain

$$V = \frac{1}{3}\pi\left(\frac{1}{2}h\right)^2 h = \frac{1}{12}\pi h^3$$

Now we can differentiate each side with respect to t:

$$\frac{dV}{dt} = \frac{1}{4}\pi h^2 \frac{dh}{dt} \implies \frac{dh}{dt} = \frac{4}{\pi h^2}\frac{dV}{dt}$$

Substituting $h = 2m$ and $\frac{dV}{dt} = \frac{1}{2}m^3/min$, we have

$$\frac{dh}{dt} = \frac{4}{\pi(2)^2}.\frac{1}{2} = \frac{1}{2\pi} \approx 0.159$$

and we see that the water level is rising at the rate of $0.159in./sec$.

Exercise 4.2

1. A passenger ship and an oil tanker left port sometime in the morning; the former headed north, and the latter headed east. At noon the passenger ship was 40 mi from port and moving at 30 mph, while the oil tanker was 30 mi from port and moving at 20 mph. How fast was the distance between the two ships changing at that time?
2. A man who is 6 ft tall walks away from a streetlight that is 15 ft from the ground at a speed of $4ft/sec$. How fast is the tip of his shadow moving along the ground when he is $30ft$ from the base of the light pole?(See figure 4.9)
3. A balloon is rising at a constant speed $4m/sec$. A boy is cycling along a straight road at a speed of $8m/sec$. When he passes under the balloon, it is 36 metres above him. How fast is the distance between the boy and balloon increasing 3 seconds later.
4. Suppose you have a cylindrical tank that has radius $10m$ and height $25m$. You pump water into the tank at a constant rate of $20m^3/min$. There is a hole in the bottom of the tank so that you lose water at a rate proportional to the height of water in the tank, say if h is the height of water, the tank looses $\frac{h}{4}m^3/min$. How fast the height of water is changing when there are $15m$ of water in the tank. Does this tank ever overflow?
5. A coffee pot that has the shape of a circular cylinder of radius $4in.$ is being filled with water flowing at a constant rate. At what rate is the water flowing into the coffee pot when the water level is rising at the rate of $0.4in./sec$? (See figure 4.10)
6. A car is traveling at 50 mph due south at a point 12 mile north of an intersection. A police car is traveling at 40 mph due west at a point 14 mile east of the same intersection. At that instant, the radar in the police car measures the rate at which the distance between the two cars is changing. What does the radar gun register?

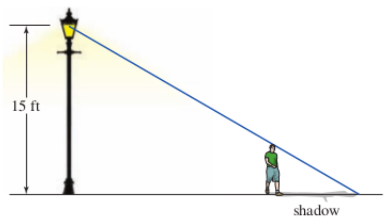


Figure 4.9:

Figure 4.10:

4.9 Derivatives of Inverse Trigonometric Functions

Definition 4.9.1 — Inverse Trigonometric Functions.

	Domain
$y = \sin^{-1} x$ iff $x = \sin y$	$[-1, 1]$
$y = \cos^{-1} x$ iff $x = \cos y$	$[-1, 1]$
$y = \tan^{-1} x$ iff $x = \tan y$	$(-\infty, \infty)$
$y = \csc^{-1} x$ iff $x = \csc y$	$(-\infty, -1] \cup [1, \infty)$
$y = \sec^{-1} x$ iff $x = \sec y$	$(-\infty, -1] \cup [1, \infty)$
$y = \cot^{-1} x$ iff $x = \cot y$	$(-\infty, \infty)$

The graphs of the six inverse trigonometric functions are shown in Figures 4.11a -4.11f

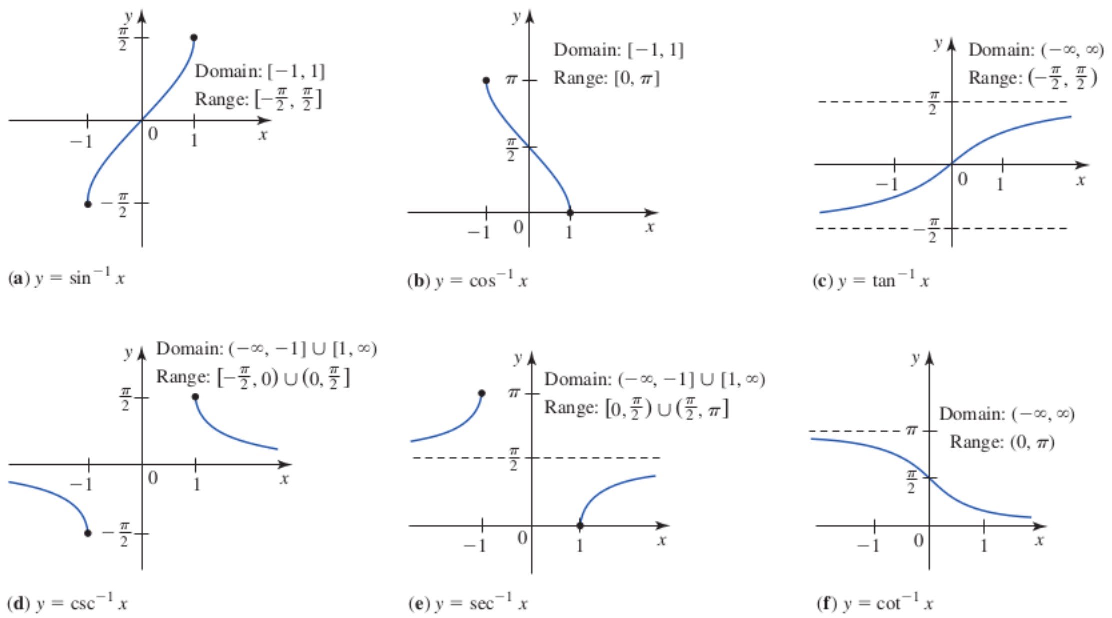


Figure 4.11:

Inverse Properties of Trigonometric Functions

$$
\begin{aligned}
\sin(\sin^{-1} x) &= x \quad \text{for} \quad -1 \le x \le 1 \\
\sin^{-1}(\sin x) &= x \quad \text{for} \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2} \\
\cos(\cos^{-1} x) &= x \quad \text{for} \quad -1 \le x \le 1 \\
\cos^{-1}(\cos x) &= x \quad \text{for} \quad 0 \le x \le \pi \\
\tan(\tan^{-1} x) &= x \quad \text{for} \quad -\infty \le x \le \infty \\
\tan^{-1}(\tan x) &= x \quad \text{for} \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}
\end{aligned}
$$

Derivatives of Inverse Trigonometric Functions

$$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$
$$\frac{d}{dx}(\cos^{-1} u) = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$
$$\frac{d}{dx}(\tan^{-1} u) = \frac{1}{1+u^2}\frac{du}{dx}$$
$$\frac{d}{dx}(\csc^{-1} u) = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$$
$$\frac{d}{dx}(\sec^{-1} u) = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$$
$$\frac{d}{dx}(\cot^{-1} u) = -\frac{1}{1+u^2}\frac{du}{dx}$$

Proof. Let $y = \sin^{-1} x \implies \sin y = x$. Differentiating implicitly with respect to x , we obtain
$$\cos y \frac{dy}{dx} = 1 \implies \frac{dy}{dx} = \frac{1}{\cos y}$$
Now, $\cos y \geq 0$, since $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, so we can write,
$$\cos y = \sqrt{1-\sin^2 y} = \sqrt{1-x^2}$$
Therefore,
$$\frac{dy}{dx} = \frac{1}{\cos y} = \frac{1}{\sqrt{1-x^2}}, \quad \text{for } -1 < x < 1$$
Finally, if u is a differentiable function of x , then the Chain Rule gives
$$\frac{d}{dx}(\sin^{-1} u) = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$
■

Example 4.31 Find the derivative of

(a) $f(x) = \cos^{-1} 5x$ (b) $g(x) = \tan^{-1}\sqrt{1+2x}$ (c) $y = \sec^{-1} e^{-2x}$

4.10 Hyperbolic Functions

Definition 4.10.1 — The Hyperbolic Functions. .

$$\sinh x = \frac{e^x - e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x}, x \neq 0 \qquad \tanh x = \frac{\sinh x}{\cosh x}$$
$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x} \qquad \coth x = \frac{\cosh x}{\sinh x}, x \neq 0$$

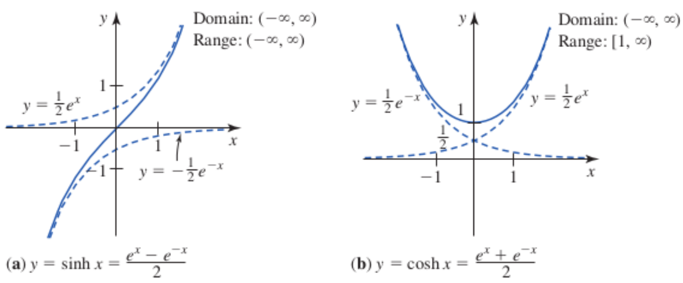


Figure 4.12: The graphs of the hyperbolic sine and cosine functions

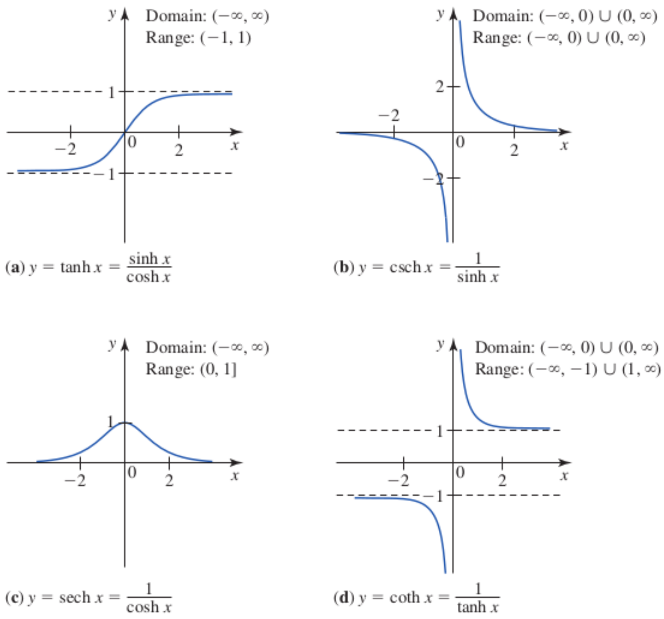


Figure 4.13: The graphs of the hyperbolic tangent, cosecant, secant, and cotangent functions

Hyperbolic Identities

$$\sinh(-x) = -\sinh x \qquad \cosh(-x) = -\cosh(x)$$
$$\cosh^2 x - \sinh^2 x = 1 \qquad \sec h^2 x = 1 - \tanh^2 x$$
$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh x \qquad \sinh 2x = 2\sinh x \cosh x$$
$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y \qquad \cosh 2x = \cosh^2 x + \sinh^2 x$$
$$\cosh^2 x = \frac{1}{2}(1+\cosh 2x) \qquad \sinh^2 x = \frac{1}{2}(-1+\cosh 2x)$$

4.11 Derivatives of Hyperbolic Functions

Derivatives of Hyperbolic Functions

$$\frac{d}{dx}(\sinh x) = \cosh x \qquad \frac{d}{dx}(\csc hx) = -\csc hx \coth x$$
$$\frac{d}{dx}(\cosh x) = \sinh x \qquad \frac{d}{dx}(\sec hx) = -\sec hx \tanh x$$
$$\frac{d}{dx}(\tanh x) = \operatorname{sech}^2 x \qquad \frac{d}{dx}(\coth x) = -\operatorname{csch}^2 x$$

■ **Example 4.32** Find the derivative of

(a) $\sinh(x^2+2)$

(b) $\cosh^2(\ln 2x)$

■

4.12 Derivatives of Inverse Hyperbolic Functions

Definition 4.12.1 — Inverse Trigonometric Functions.

	Domain
$y = \sinh^{-1} x$ iff $x = \sinh y$	$(-\infty, \infty)$
$y = \cosh^{-1} x$ iff $x = \cosh y$	$[1, \infty)$
$y = \tanh^{-1} x$ iff $x = \tanh y$	$(-1, 1)$
$y = \operatorname{csch}^{-1} x$ iff $x = \csc hy$	$(-\infty, 0) \cup (0, \infty)$
$y = \operatorname{sech}^{-1} x$ iff $x = \sec hy$	$(0, 1]$
$y = \coth^{-1} x$ iff $x = \coth y$	$(-\infty, -1) \cup (1, \infty)$

■ **Example 4.33** Show that $\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$ ■

Solution: Let $y = \sinh^{-1} x$. Then

$$x = \sinh y = \frac{e^y - e^{-y}}{2} \tag{4.7}$$

$$\Longrightarrow \quad e^y - 2x - e^{-y} = 0 \Longrightarrow e^{2y} - 2xe^y - 1 = 0 \tag{4.8}$$

which is a quadratic in e^y . Using the quadratic formula, we have

$$e^y = x \pm \sqrt{x^2+1}$$

Only the root $x + \sqrt{x^2+1}$ is admissible. To see why, observe that $e^y > 0$, but $x - \sqrt{x^2+1} < 0$ since $x < \sqrt{x^2+1}$. Therefore, we have

$$e^y = x + \sqrt{x^2+1}$$

so

$$y = \ln(x + \sqrt{x^2+1})$$

That is

$$\sinh^{-1} x = \ln(x + \sqrt{x^2+1})$$

Representations of Inverse Hyperbolic Functions in Terms of Logarithmic Functions

$$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}) \qquad (\infty, \infty)$$

$$\cosh^{-1} x = \ln(x + \sqrt{x^2-1}) \qquad [1, \infty)$$

$$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \qquad (-1, 1)$$

The derivatives of the inverse hyperbolic functions can be found by differentiating the function in question directly. For example,

$$\begin{aligned}
\frac{d}{dx}(\cosh^{-1} x) &= \frac{d}{dx}(\ln(x + \sqrt{x^2-1}) \\
&= \frac{1}{x + \sqrt{x^2-1}}\left(1 + \frac{x}{\sqrt{x^2-1}}\right) \\
&= \frac{1}{x + \sqrt{x^2-1}}\left(\frac{\sqrt{x^2-1} + x}{\sqrt{x^2-1}}\right) \\
&= \frac{1}{\sqrt{x^2-1}}
\end{aligned}$$

Derivatives of Inverse Hyperbolic Functions

$$\frac{d}{dx}(\sinh^{-1}u) = \frac{1}{\sqrt{u^2+1}}\frac{du}{dx}$$
$$\frac{d}{dx}(\cosh^{-1}u) = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}$$
$$\frac{d}{dx}(\tanh^{-1}u) = \frac{1}{1-u^2}\frac{du}{dx}$$
$$\frac{d}{dx}(\text{csch}^{-1}u) = -\frac{1}{|u|\sqrt{u^2+1}}\frac{du}{dx}$$
$$\frac{d}{dx}(\text{sech}^{-1}u) = -\frac{1}{u\sqrt{1-u^2}}\frac{du}{dx}$$
$$\frac{d}{dx}(\coth^{-1}u) = \frac{1}{1-u^2}\frac{du}{dx}$$

■ **Example 4.34** Find the derivative of $y = x^2\text{sech}^{-1}3x$ ■

4.13 Applications of Derivatives

4.13.1 Extrema of a Function f

Definition 4.13.1 — Extrema of a Function f.
A function f has an absolute maximum at c if $f(x) \leq f(c)$ for all x in the domain D of f. The number $f(c)$ is called the maximum value of f on D. Similarly, f has an absolute minimum at c if $f(x) \geq f(c)$ for all x in D. The number $f(c)$ is called the minimum value of f on D. The absolute maximum and absolute minimum values of f on D are called the extreme values, or extrema, of f on D.

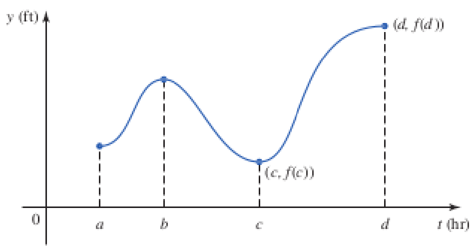


Figure 4.14:

Definition 4.13.2 — Relative Extrema of a Function.
A function f has a relative (or local) maximum at c if $f(c) \geq f(x)$ for all values of x in some open interval containing c. Similarly, f has a relative (or local) minimum at c if $f(c) \leq f(x)$ for all values of x in some open interval containing c.

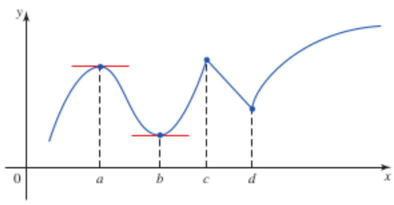


Figure 4.15: The function f has relative extrema at a, b, c, and d.
The tangent lines at a and b are horizontal. There are no tangent lines at c and d.

Theorem 4.13.1 — Fermat's Theorem.
If f has a relative extremum at c, then either $f'(c) = 0$ or $f'(c)$ does not exist.

Definition 4.13.3 — Critical Number of f.
A critical number of a function f is any number c in the domain of f at which $f'(c) = 0$ or $f'(c)$ does not exist.

■ **Example 4.35** Find the critical numbers of $f(x) = x - 3\sqrt[3]{x}$ ■

Solution: $f'(x) = \dfrac{x^{2/3} - 1}{x^{2/3}}$. Then the critical numbers are $-1, 0,$ and 1.

Theorem 4.13.2 — The Extreme Value Theorem.
If f is continuous on a closed interval $[a,b]$, then f attains an absolute maximum value $f(c)$ for some number c in $[a,b]$ and an absolute minimum value $f(d)$ for some number d in $[a,b]$.

Guidelines for Finding the Extrema of a Continuous Function f on $[a,b]$

1. Find the critical numbers of f that lie in (a,b).
2. Compute the value of f at each of these critical numbers, and also compute $f(a)$ and $f(b)$.
3. The absolute maximum value of f and the absolute minimum value of f are precisely the largest and the smallest numbers found in Step 2.

■ **Example 4.36** Find the extreme values of the function

(a) $f(x) = 3x^4 - 4x^3 - 8$ on $[-1,2]$

(b) $f(x) = 2cosx - x$ on $[0, 2\pi]$ (c) $f(x) = \dfrac{1}{x}$ on $[1,3]$ ■

Solution:

(a) Since f is a polynomial function, it is continuous everywhere; in particular, it is continuous on the closed interval $[-1,2]$. Therefore, we can use the Extreme Value Theorem.
First, we find the critical numbers of f in $(-1,2)$:
$f'(x) = 12x^3 - 12x^2 = 12x^2(x-1)$
Observe that $f'(x)$ is continuous on $(-1,2)$. Next, setting $f'(x) = 0$ gives $x = 0$ or $x = 1$. Therefore, 0 and 1 are the only critical numbers of f in $(-1,2)$.
Next, we compute $f(x)$ at these critical numbers as well as at the endpoints -1 and 2. These values are $f(0) = -8, f(1) = -9, f(-1) = -1$ and $f(2) = 8$
Thus, f attains the absolute maximum value of 8 at 2 and the absolute minimum value of -9 at 1.

(b) The function f is continuous everywhere; in particular, it is continuous on the closed interval $[0, 2\pi]$. Therefore, the Extreme Value Theorem is applicable. First, we find the critical numbers of f in $(0, 2\pi)$. We have $f'(x) = -2sinx - 1 = 0$
Observe that $f'(x)$ is continuous on $(0, 2\pi)$. Next, setting $f'(x) = 0$ gives
$$-2\sin x - 1 = 0 \Rightarrow = \frac{-1}{2}$$
Thus, $x = \dfrac{7\pi}{6}$ or $\dfrac{11\pi}{6}$. (Remember x lies in $(0, 2\pi)$.) So $\dfrac{7\pi}{6}$ and $\dfrac{11\pi}{6}$ are the only critical numbers of f in $(0, 2\pi)$.
Next, we compute the values of f at these critical numbers as well as at the endpoints 0 and 2π. These values are

$$f(0) = 2, f(2\pi) = 2 - 2\pi = -4.28, f(\frac{7\pi}{6}) = -\sqrt{3} - \frac{7\pi}{6} = -5.40,$$
$$f(\frac{11\pi}{6}) = \sqrt{3} - \frac{7\pi}{6} = -4.03$$

Thus, f attains the absolute maximum value of 2 at 0 and the absolute minimum value of approximately -5.4 at $\frac{7\pi}{6}$.

(c) Notice that on the interval $[1,3]$, f is continuous. Consequently, the Extreme Value Theorem guarantees that f has both an absolute maximum and an absolute minimum on $[1,3]$. Hence, $f(x)$ reaches its maximum value of 1 at $x = 1$ and its minimum value of $\frac{1}{3}$ at $x = 3$.

4.13.2 The Mean Value Theorem

Theorem 4.13.3 Rolle's Theorem
Let f be continuous on $[a,b]$ and differentiable on (a,b). If $f(a) = f(b)$, then there exists at least one number c in (a,b) such that $f'(c) = 0$.

Example 4.37 Let $f(x) = x^3 - x$
(a) Show that f satisfies the hypotheses of Rolle's Theorem on $[-1,1]$.
(b) Find the number(s) c in $(-1,1)$ such that $f'(c) = 0$ as guaranteed by Rolle's Theorem.

Solution:

(a) The polynomial function f is continuous and differentiable on $(-\infty,\infty)$. In particular, it is continuous on $[-1,1]$ and differentiable on $(-1,1)$. Furthermore,
$$f(-1) = (-1)^3 - (-1) = 0 \text{ and } f(1) = 1^3 - 1 = 0$$
and the hypotheses of Rolle's theorem are satisfied.

(b) Rolle's Theorem guarantees that there exists at least one number c in $(-1,1)$ such that $f'(c) = 0$. But $f'(x) = 3x^2 - 1$, so to find c , we solve
$$3c^2 - 1 = 0 \implies c = \pm\frac{\sqrt{3}}{3}$$

Theorem 4.13.4 The Mean Value Theorem
Let f be continuous on $[a,b]$ and differentiable on (a,b). Then there exists at least one number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{4.9}$$

Example 4.38 Let $f(x) = x^3 - 5x^2 - 3x$
(a) Show that f satisfies the hypothesis of the Mean Value Theorem on $[1,3]$.
(b) Find the number(s) c in $(1,3)$ such that $f'(c) = \frac{f(3) - f(1)}{3 - 1}$ as guaranteed by the Mean Value Theorem.

Solution:

(a) f is a polynomial function, so it is continuous and differentiable on $(-\infty,\infty)$. In particular, it is continuous on $[-1,1]$ and differentiable on $(-1,1)$. So the hypotheses of the Mean Value Theorem are satisfied.

(b) $f'(x) = 3x^2 - 10x - 3$, so $f'(c) = 3c^2 - 10c - 3$. With $a = 1$ and $b = 3$, Equation (4.9) gives

$$\begin{aligned} f'(c) &= \frac{f(3) - f(1)}{3-1} \\ 3c^2 - 10c - 3 &= \frac{-27 - (-7)}{3-1} \\ 3c^2 - 10c - 3 &= -10 \implies (3c-7)(c-1) \\ c = 1 \text{ or } c = \frac{7}{3} \end{aligned}$$

So there are two numbers, $c_1 = 1$ and $c_2 = \frac{7}{3}$, in $(1,3)$ that satisfy Equation (4.9).

4.13.3 Increasing and Decreasing Functions and The First Derivative Test

Definition 4.13.4 Increasing and Decreasing Functions
A function f is increasing on an interval I, if for every pair of numbers x_1 and x_2 in I,

$$x_1 < x_2 \text{ implies that } f(x_1) < f(x_2)$$

f is decreasing on I if, for every pair of numbers x_1 and x_2 in I,

$$x_1 < x_2 \text{ implies that } f(x_1) > f(x_2)$$

f is monotonic on I if it is either increasing or decreasing on I.

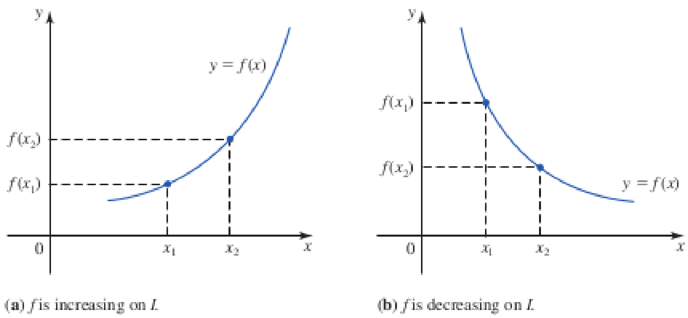


Figure 4.16:

Theorem 4.13.5 Suppose f is differentiable on an open interval (a,b).
1. If $f'(x) > 0$ for all x in (a,b), then f is increasing on (a,b).
2. If $f'(x) < 0$ for all x in (a,b), then f is decreasing on (a,b).
3. If $f'(x) = 0$ for all x in (a,b), then f is constant on (a,b).

Determining the Intervals Where a Function Is Increasing or Decreasing
1. Find all the values of x for which $f'(x) = 0$ or $f'(x)$ does not exist. Use these values of x to partition the domain of f into open intervals.
2. Select a test number c in each interval found in Step 1, and determine the sign of $f'(c)$ in that interval.
   (a) If $f'(c) > 0$, then f is increasing on that interval.
   (b) If $f'(c) < 0$, then f is decreasing on that interval.
   (c) If $f'(c) = 0$, then f is constant on that interval.

Example 4.39 Determine the intervals where the function $f(x) = x^3 + x^2 - 5x - 5$ increasing and where it is decreasing. ■

Solution: $$f'(x) = 3x^2 + 2x - 5 = (x-1)(3x+5)$$
from which we see that f' is continuous everywhere and has zeros at $-\frac{5}{3}$ and 1. These zeros of f' partition the domain of f into the intervals $(-\infty, -\frac{5}{3}), (-\frac{5}{3}, 1)$, and $(1, \infty)$.
To determine the sign of $f'(x)$ on each of these intervals, we evaluate $f'(x)$ at a convenient test number in each interval. These results are summarized in the following table.

Interval	k	Test value $f'(k)$	sign of $f'(x)$	variation of f
$\left(-\infty, -\frac{5}{3}\right)$	-2	3	+	increasing on $\left(-\infty, -\frac{5}{3}\right]$
$\left(-\frac{5}{3}, 1\right)$	0	-5	-	decreasing on $\left[-\frac{5}{3}, 1\right]$
$(1, \infty)$	2	11	+	increasing on $(1, \infty)$

We conclude that f is increasing on $\left(-\infty, -\frac{5}{3}\right)$ and $(1, \infty)$ and decreasing on $\left[-\frac{5}{3}, 1\right]$.

Example 4.40 Determine the intervals where the function $f(x) = x + \frac{1}{x}$ is increasing and where it is decreasing. ■

Solution: The derivative of f is
$$f'(x) = \frac{(x+1)(x-1)}{x^2}$$
from which we see that $f'(x)$ is continuous everywhere except at $x = 0$ and has zeros at $x = -1$ and $x = 1$. These values of x partition the domain of f into the intervals $(-\infty, -1), (-1, 0), (0, 1)$, and $(1, \infty)$.
By evaluating $f'(x)$ at each of the test numbers$x = -2, -\frac{1}{2}, \frac{1}{2}$, and 2, we find
$f'(-2) = \frac{3}{4}, f'(-0.5) = -3, f'(0.5) = -3$, and $f'(2) = \frac{3}{4}$. We conclude that f is increasing on $(-\infty, -1)$ and $(1, \infty)$ and decreasing on $(-1, 0)$ and $(0, 1)$. Note that $f'(x)$ does not change sign as we move across the point of discontinuity.

Theorem 4.13.6 The First Derivative Test
Let c be a critical number of a continuous function f in the interval (a, b) and suppose that f is differentiable at every number in (a, b) with the possible exception of c itself.

1. If $f'(x) > 0$ on (a, c) and $f'(x) < 0$ on (c, b), then f has a relative maximum at c
2. If $f'(x) < 0$ on (a, c) and $f'(x) > 0$ on (c, b), then f has a relative minimum at c
3. If $f'(x)$ has the same sign on (a, c) and (c, b), then f does not have a relative extremum at c

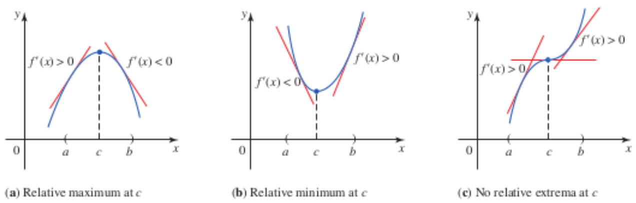


Figure 4.17:

Example 4.41 Find the relative extrema of $f(x) = x^4 - 4x^3 + 12$

Solution: The derivative of f,

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x-3)$$

is continuous everywhere. Therefore, the zeros of f', which are 0 and 3, are the only critical numbers of f. The sign diagram of f' is shown in Figure 4.18. Since f' has the same sign on $(-\infty, 0)$ and $(0, 3)$, the First Derivative Test tells us that f does not have a relative extremum at 0. Next, we note that f' changes sign from negative to positive as we move across 3, so 3 does give rise to a relative minimum of f. The relative minimum value of f is $f(3) = -15$. The graph of f is shown in Figure 4.19 and confirms these results.

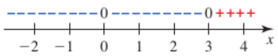


Figure 4.18: The sign diagram of f'

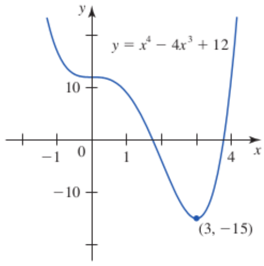


Figure 4.19: The graph of f

Example 4.42 Find the local extrema of f if $f(x) = x^{2/3}(x^2 - 8)$.

Solution: The derivatives of f

$$f'(x) = \frac{8(x^2-2)}{3x^{1/3}}.$$

The critical numbers are $-\sqrt{2}, 0,$ and $\sqrt{2}$. These suggest an examination of the sign of $f'(x)$ in the intervals $(-\infty, -\sqrt{2}), (-\sqrt{2}, 0), (0, \sqrt{2})$ and $(\sqrt{2}, \infty)$.

Interval	k	Test value $f'(k)$	sign of $f'(x)$	variation of f
$\left(-\infty, -\sqrt{2}\right)$	-8	$-\frac{248}{3}$	-	decreasing on $\left(-\infty, -\sqrt{2}\right]$
$\left(-\sqrt{2}, 0\right)$	-1	$\frac{8}{3}$	+	increasing on $\left[-\sqrt{2}, 0\right]$
$(0, \sqrt{2})$	1	$-\frac{8}{3}$	-	decreasing on $(0, \sqrt{2})$
$(\sqrt{2}, \infty)$	8	$\frac{248}{3}$	+	increasing on $(\sqrt{2}, \infty)$

By the First Derivative Test, f has local minima at $-\sqrt{2}$ and $\sqrt{2}$ and a local minima at 0.

Example 4.43 Find the relative extrema of $x^3 - 6x^2 + 9x + 1$

4.13.4 Concavity and Inflection Point

Definition 4.13.5 Concavity of the Graph of a Function
Suppose f is differentiable on an open interval I. Then
1. the graph of f is concave upward on I if f' is increasing on I.
2. the graph of f is concave downward on I if f' is decreasing on I.

Theorem 4.13.7 Suppose f has a second derivative on an open interval I.
1. If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.
2. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Definition 4.13.6 — Inflection Point. Let the function f be continuous on an open interval containing the point c, and suppose the graph of f has a tangent line at $P(c, f(c))$. If the graph of f changes from concave upward to concave downward (or vice versa) at P, then the point P is called an inflection point of the graph of f.

Theorem 4.13.8 The Second Derivative Test
Suppose that f has a continuous second derivative on an interval (a,b) containing a critical number c of f.
1. If $f''(c) > 0$, then f has a relative minimum at c.
2. If $f''(x) < 0$, then f has a relative maximum at c.
3. If $f''(x) = 0$, then the test is inconclusive.

■ **Example 4.44** Find the relative extrema of $f(x) = x^4 + \frac{4}{3}x^3 - x^2$ using the Second Derivative Test. ■

Solution: $f'(x) = 4x^3 + 4x^2 - 2x$.
Setting $f'(x) = 0$, we see that 0, $\frac{-1-\sqrt{3}}{2}$ and $\frac{-1+\sqrt{3}}{2}$ are critical numbers of f. Next, we compute

$$f''(x) = 12x^2 + 8x - 2$$

Evaluating $f''(x)$ at the critical number 0, we find

$$f''(0) = -2 < 0$$

and the Second Derivative Test implies that 0 gives rise to a relative maximum of f. Also

$$f''\left(\frac{-1-\sqrt{3}}{2}\right) => 0 \text{ and } \qquad f''\left(\frac{-1+\sqrt{3}}{2}\right) => 0$$

so $\frac{-1\pm\sqrt{3}}{2}$ gives rise to a relative minimum of f.

4.13.5 Curve Sketching

Guidelines for Sketching a Curve
1. Find the domain of f.
2. Find the x- and y-intercepts of f.
3. Determine whether the graph of f is symmetric with respect to the y-axis or the origin. That is
   (a) If $f(-x) = f(x)$ for all x in the domain D, then f is an even function and the curve is symmetric about the y-axis.
   (b) If $f(-x) = -f(x)$ for all x in the domain D ,then f is an odd function and the curve is symmetric about the origin.
4. Determine the behavior of f for large absolute values of x.
5. Find the asymptotes of the graph of f.
6. Find the intervals where f is increasing and where f is decreasing.
7. Find the relative extrema of f.
8. Determine the concavity of the graph of f.
9. Find the inflection points of f.
10. Sketch the graph of f

Example 4.45 Let $y = \dfrac{x^2}{x^2-1}$. Then sketch the graph of y.

Solution:

1. The denominator of f is $(-\infty,\ -1)\cup(1,\ 1)\cup(1,\ \infty)$.
2. Setting $x=0$ gives 0 as the y -intercept. Next, setting $f(x)=0$ gives $x^2=0$, or $x=0$. So the x -intercept is 0.
3. $f(-x) = \dfrac{(-x)^2}{(-x)^2-1} = \dfrac{x^2}{x^2-1} = f(x)$ and this shows that the graph of f is symmetric with respect to the y-axis.
4. $\lim\limits_{x\to-\infty} \dfrac{x^2}{x^2-1} = \lim\limits_{x\to\infty} \dfrac{x^2}{x^2-1} = 1$
5. Because the denominator of $f(x)$ is equal to zero at -1 and 1, the lines $x=-1$ and $x=1$ are candidates for the vertical asymptotes of the graph of f. Since
$$\lim_{x\to-1^-} \frac{x^2}{x^2-1} = \infty \text{ and } \lim_{x\to1^-} \frac{x^2}{x^2-1} = -\infty$$
we see that $x=-1$ and $x=1$ are indeed vertical asymptotes. From part (4) we see that $y=1$ is a horizontal asymptote of the graph of f.
6. $f'(x) = -\dfrac{2x}{(x^2-1)^2}$
Notice that f' is continuous everywhere except at ±1 and that it has a zero when $x=0$. The sign diagram of f' is shown in Figure 4.20.
From the diagram we see that f is increasing on $(-\infty,-1)$ and on $(-1,0)$ and decreasing on

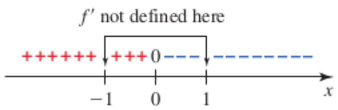


Figure 4.20:

$(0,1)$ and on $(1,\infty)$.
7. From the results of part (6) we see that 0 is a critical number of f. The numbers -1 and 1 are not in the domain of f and, therefore, are not critical numbers of f.
Also, from Figure 4.20 we see that f has a relative maximum at $x=0$. Its value is $f(0)=0$.
8. $f''(x) = \dfrac{2(3x^2+1)}{(x^2-1)^3}$
Notice that f'' is continuous everywhere except at ±1 and that f'' has no zero. The sign diagram of f'' is shown in Figure 4.21, we see that the graph of f is concave upward on $(-\infty,-1)$ and on $(1,\infty)$ and concave downward on $(-1,1)$.

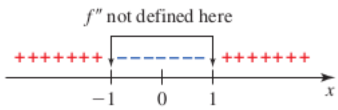


Figure 4.21:

9. f has no inflection points. Remember that -1 and 1 are not in the domain of f.

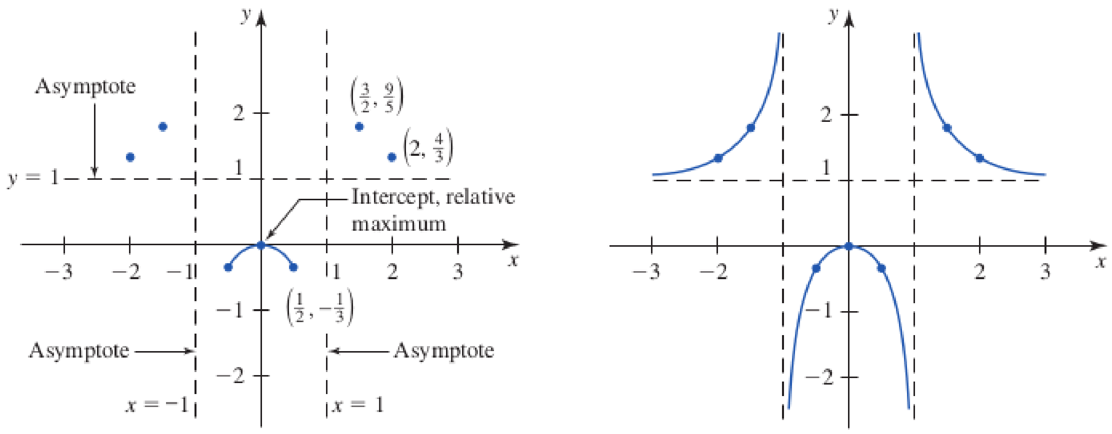


Figure 4.22:

Slant Asymptotes

The graph of a function f may have an asymptote that is neither vertical nor horizontal but slanted. We call the line with equation $y = mx + b$ a **slant** or **oblique** (right) asymptote of the graph of f if

$$\lim_{x\to\infty} \frac{f(x)}{x} = m \quad \text{and} \quad \lim_{x\to\infty} (f(x) - mx) = b$$

4.14 Indeterminate Forms and l'Hospital's Rule

4.14.1 The Indeterminate Form 0/0 and ∞/∞

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = 0$, then the limit

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

is called an **indeterminate form of the type** $0/0$. As the name implies, the undefined expression $0/0$ does not provide us with a definitive answer concerning the existence of the limit or its value, if the limit exists.

If $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$, then the limit

$$\lim_{x\to a} \frac{f(x)}{g(x)}$$

is called an **indeterminate form of the type** ∞/∞

To see why this limit is an indeterminate form, we simply write

$$\lim_{x\to a} \frac{\frac{1}{g(x)}}{\frac{1}{f(x)}}$$

which has the form $0/0$ and, therefore, is indeterminate.

Theorem 4.14.1 — l'Hôpital's Rule. Suppose that f and g are differentiable on an open interval I that contains a , with the possible exception of a itself, and $g'(x) \neq 0$ for all x in I. If $\lim_{x\to a} \frac{f(x)}{g(x)}$ is indeterminate form of the type $0/0$ or ∞/∞, then

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)}$$

provided that the limit on the right-hand side exists or is infinite.

R l'Hôpital's Rule is also valid for one-sided limits as well as limits at infinity or negative infinity; that is, we can replace "$x \to a$" by any of the symbols $x \to a^+, x \to a^-$, $x \to \infty$, or $x \to -\infty$.

Example 4.46 Evaluate

(a) $\lim_{x\to 0} \frac{\sin 3x}{\sin 2x}$

(b) $\lim_{x\to\infty} \frac{\ln x}{x}$

(c) $\lim_{x\to 1^+} \frac{\sin \pi x}{\sqrt{x-1}}$

(d) $\lim_{x\to 0} \frac{e^x - x - 1}{x^2}$

(e) $\lim_{x\to\infty} \frac{x^3}{e^{2x}}$

(f) $\lim_{x\to 0} \frac{x^3}{x - \tan x}$

4.14.2 The Indeterminate Form $\infty - \infty$ and $0.\infty$

If $\lim_{x\to a} f(x) = \infty$ and $\lim_{x\to a} g(x) = \infty$, then the limit

$$\lim_{x\to a}(f(x) - g(x))$$

is said to be an **indeterminate form of the type** $\infty - \infty$. An indeterminate form of this type can often be expressed as one of the type $0/0$ or ∞/∞ by algebraic manipulation.

If $\lim_{x\to a} f(x) = 0$ and $\lim_{x\to a} g(x) = \pm\infty$, then the limit

$$\lim_{x\to a}(f(x)g(x))$$

is said to be an **indeterminate form of the type** $0.\infty$. An indeterminate form of this type can often be expressed as one of the type $0/0$ or ∞/∞ by algebraic manipulation.

Example 4.47 Evaluate

(a) $\lim_{x\to 0^+} \left(\frac{1}{x} - \frac{1}{e^x - 1}\right)$

(b) $\lim_{x\to 0} \left(\frac{1}{\ln(x+1)} - \frac{1}{x}\right)$

(c) $\lim_{x\to 0^+} x \ln x$

(d) $\lim_{x\to\infty} \frac{\ln x}{x}$

4.14.3 The Indeterminate Form $\infty - \infty$ and $0.\infty$

The limit

$$\lim_{x\to a}[f(x)]^{g(x)}$$

is said to be an **indeterminate form of the type**

$$0^0 \text{ if } \lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = 0$$

$$\infty^0 \text{ if } \lim_{x\to a} f(x) = \infty \text{ and } \lim_{x\to a} g(x) = 0$$

$$1^0 \text{ if } \lim_{x\to a} f(x) = 1 \text{ and } \lim_{x\to a} g(x) = \infty$$

These indeterminate forms can usually be converted to indeterminate forms of the type $0.\infty$ by taking logarithms or by using the identity

$$[f(x)]^{g(x)} = e^{g(x)\ln f(x)}$$

Example 4.48 Evaluate

(a) $\lim_{x\to 0^+} x^x$ (b) $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x$ (c) $\lim_{x\to 1^+} x^{\frac{1}{x-1}}$